

# On $S$ -duality in $(2 + 1)$ -Chern-Simons Supergravity

H. García-Compeán<sup>a\*</sup>, O. Obregón<sup>b†</sup>, C. Ramírez<sup>c‡</sup> and M. Sabido<sup>b§</sup>

<sup>a</sup> *Departamento de Física,*

*Centro de Investigación y de Estudios Avanzados del IPN*

*P.O. Box 14-740, 07000, México D.F., México*

<sup>b</sup> *Instituto de Física de la Universidad de Guanajuato*

*P.O. Box E-143, 37150, León Gto., México*

<sup>c</sup> *Facultad de Ciencias Físico Matemáticas, Universidad Autónoma de Puebla*

*P.O. Box 1364, 72000, Puebla, México*

(February 1, 2008)

## Abstract

Strong/weak coupling duality in Chern-Simons supergravity is studied. It is argued that this duality can be regarded as an example of superduality. The use of supergroup techniques for the description of Chern-Simons supergravity greatly facilitates the analysis.

---

\*E-mail address: `compean@fis.cinvestav.mx`

†E-mail address: `octavio@ifug3.ugto.mx`

‡E-mail address: `cramirez@fcfm.buap.mx`

§E-mail address: `msabido@ifug3.ugto.mx`

## I. INTRODUCTION

Very recently a great deal of work has been pursued in the realization of a duality in the abelian and non-abelian gauge theory with and without supersymmetry (for a review see [1–3]). A very interesting gauge theory in  $(2 + 1)$  dimensions which contains many topological properties is Chern-Simons gauge theory. In particular, this theory describes knots and links invariants [4].

Duality properties in abelian Chern-Simons gauge theory were worked out in Ref. [5]. This duality has potential applications to the fractional quantum Hall effect and other lower dimensional condensed matter systems [6].

However this duality is only well defined up to some control over the running coupling constant. This control is provided by imposing a flux quantization condition [5], or when certain degree of supersymmetry is present. For instance, for the  $\mathcal{N} = 3$  Chern-Simons QED, whose coupling constant is marginal, the mirror symmetry behaves as the strong/weak coupling duality inverting the coupling constant  $k \leftrightarrow -\frac{1}{k}$  [7].

Duality in non-abelian Chern-Simons gauge theory is still missing in the literature and some attempts to define it were done in Ref. [8] in the context of  $(2 + 1)$  -Chern-Simons gravity, by using an alternative description of the Achúcarro-Townsend and Witten description [9,10].

On the other hand, it has been shown that dual theories to a topological gravitational model [11], to the MacDowell-Mansouri (MM) gauge theories of gravity [12] and to supergravity [13], can be constructed. These dual theories result in non-linear  $\sigma$ -models of the Freedman-Townsend type. In these examples, the duality is performed on the gauge group which is the Lorentz group  $SO(3,1)$ . Thus gravitational duality involves a duality in the spacetime symmetry group. Spacetime duality has also been defined in 1+1 dimensions in Ref. [14].

Duality in Chern-Simons gravity considered in Ref. [8] is precisely another example of spacetime duality [11–14]. The variables of this theory are the gauge connection  $A_i^{AB}$  which

contains the dreibein  $e_i^a$  and the spin connection  $\omega_i^{ab}$ . In the spacetime duality algorithm, the gauged symmetry for this case is the gauge symmetry  $SO(2,2)$  with self-dual  $SL(2, \mathbf{R})$  and anti-self-dual  $SL(2, \mathbf{R})$  components. The constraints are implemented through a Lagrange multiplier field  $\Lambda$ . Then integrating out with respect to  $\Lambda$  we recover the original Chern-Simons gravity Lagrangian but integrating out with respect to the gravitational variables  $A_i^{AB}$  or  $(e_i^a, \omega_i^{ab})$  we get its dual Lagrangian. Thus, given the central role of supersymmetry in the study of duality, it is natural to look for extending this spacetime duality to *superduality* of Refs. [13,14]. This is the aim of the present paper. We will show that this [14] superduality can be realized by the introduction of (2+1) Chern-Simons supergravity theory as formulated in Ref. [15] through the gauging of the supergroup  $OSp(2, 2|1)$ . Actually the  $S$ -duality of MacDowell-Mansouri supergravity theory found in Ref. [13] is an example of superduality.

## II. CHERN-SIMONS SUPERGRAVITY FROM THE SELF-DUAL SPIN SUPERCONNECTION

We will work out the Chern-Simons Lagrangian for (anti) self-dual spin superconnection with respect to duality transformations of the internal indices, in the same philosophy of [8,16]. To do that we first introduce some notation of Chern-Simons supergravity for future uses (see also Refs. [15]). For definiteness let us take the gauge group  $SO(2, 2)$ , with metric  $\eta^{AB} = \text{diag}(-1, -1, 1, 1)$ , where the indices  $A, B$  run from 0 to 3 and whose corresponding supergroup is generated by the superalgebra which we will call  $OSp(2, 2|1)$

$$\begin{aligned} [M_{AB}, M_{CD}] &= \frac{1}{2}(\eta_{AC}M_{DB} + \eta_{BD}M_{CA} - \eta_{BC}M_{DA} - \eta_{AD}M_{CB}), \\ [M_{AB}, Q_\alpha] &= \frac{1}{2}(\gamma_{AB})_{\alpha\beta}Q_\beta, \\ \{Q_\alpha, Q_\beta\} &= -\frac{1}{2}(\gamma^{AB}C)_{\alpha\beta}M_{AB}, \end{aligned} \tag{1}$$

where  $M_{AB}$  and  $Q_\alpha$  are the generators of this algebra.

Our conventions for the Dirac matrices are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & \tau^a \\ -\tau^a & 0 \end{pmatrix}, \tag{2}$$

where  $\tau^1 = i\sigma^2, \tau^2 = \sigma^1, \tau^3 = \sigma^3$ . Further  $\gamma^{AB} = \frac{1}{4}[\gamma^A, \gamma^B]$ ,  $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$  and the charge conjugation matrix satisfies  $C\gamma^AC^{-1} = -\gamma^{AT}$ ,  $C^T = -C$  and it is given by

$$C = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}. \quad (3)$$

Thus, the Chern-Simons action for a  $Osp(2, 2|1)$  algebra-valued gauge connection  $\mathbf{A}_i$  on the three-dimensional manifold  $X$  is given by the usual expression

$$L_{SCS} = STr \int_X d^3x \varepsilon^{ijk} \left( \mathbf{A}_i \partial_j \mathbf{A}_k + \frac{2}{3} \mathbf{A}_i \mathbf{A}_j \mathbf{A}_k \right), \quad (4)$$

where  $STr$  is the supertrace (for notation and terminology see [17]),

$$\mathbf{A}_i = \mathbf{A}_i^{\mathcal{A}} M_{\mathcal{A}} \equiv A_i^{AB} M_{AB} + A_i^{\alpha} Q_{\alpha}, \quad (5)$$

and the generators  $M_{\mathcal{A}}$  are in the adjoint representation. Thus we define

$$\eta_{AB} = STr(M_{\mathcal{A}} M_{\mathcal{B}}) = \text{diag}(\eta_{AB,CD}, C_{\alpha\beta}), \quad (6)$$

is the Cartan metric,  $\eta_{AB,CD} = -\eta_{AC}\eta_{BD} + \eta_{BC}\eta_{AD}$  and  $C$  is the charge conjugation matrix.

Thus, the action (4) can be rewritten as

$$\begin{aligned} L_{SCS} &= \int_X d^3x \varepsilon^{ijk} \left( \mathbf{A}_i^{\mathcal{A}} \partial_j \mathbf{A}_k^{\mathcal{B}} \eta_{\mathcal{B}\mathcal{A}} + \frac{2}{3} \mathbf{A}_i^{\mathcal{A}} \mathbf{A}_j^{\mathcal{B}} \mathbf{A}_k^{\mathcal{C}} STr(M_{\mathcal{C}}[M_{\mathcal{B}}, M_{\mathcal{A}}]) \right) \\ &= \int_X d^3x \varepsilon^{ijk} \left( \mathbf{A}_i^{\mathcal{A}} \partial_j \mathbf{A}_k^{\mathcal{B}} \eta_{\mathcal{B}\mathcal{A}} - \frac{1}{3} f_{\mathcal{B}\mathcal{A}}^{\mathcal{D}} \eta_{\mathcal{C}\mathcal{D}} \mathbf{A}_i^{\mathcal{A}} \mathbf{A}_j^{\mathcal{B}} \mathbf{A}_k^{\mathcal{C}} \right). \end{aligned} \quad (7)$$

This action can be written as

$$\begin{aligned} L_{SCS} &= \int_X d^3x \varepsilon^{ijk} \left( A_i^{AB} \partial_j A_{kBA} + \frac{2}{3} A_{iA}^B A_{jB}^C A_{kC}^A \right) \\ &+ \int_X d^3x \varepsilon^{ijk} \left( A_i^{\alpha} \partial_j A_k^{\beta} C_{\alpha\beta} + \frac{1}{2} (\gamma_{AB} C)_{\alpha\beta} A_i^{AB} A_j^{\alpha} A_k^{\beta} \right), \end{aligned} \quad (8)$$

where  $f_{AB}^{\mathcal{C}}$  are structure constants of the superalgebra (1).

It is well known that the group  $SO(2, 2)$  is decomposed as

$$SO(2, 2) = SL(2, \mathbf{R}) \times SL(2, \mathbf{R}), \quad (9)$$

thus, the spinors are real and are decomposed into two  $SL(2, \mathbf{R})$  two-component spinors

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \phi_\beta \end{pmatrix}. \quad (10)$$

Moreover, if the Majorana condition  $\bar{\Psi} = -\Psi^T C^{-1}$  is imposed, then  $\phi_\alpha = -\varepsilon_{\alpha\beta} \psi_\beta$ , where  $\varepsilon = i\sigma^2$ .

The decomposition of the group (9), can be traced back to the decomposition of the generators  $M$  and  $Q$  into self-dual and anti-self-dual parts as follows

$$\begin{aligned} M^\pm_{AB} &= \frac{1}{2} \left( M_{AB} \pm \frac{1}{2} \epsilon_{AB}{}^{CD} M_{CD} \right), \\ Q^\pm_\alpha &= \frac{1}{2} \left( Q_\alpha \pm (\gamma^5 Q)_\alpha \right). \end{aligned} \quad (11)$$

Indeed, it is an easy matter to show that  $M^\pm$  and  $Q^\pm$  satisfy the same algebra as  $M$  and  $Q$ ,

$$\begin{aligned} [M^\pm_{AB}, M^\pm_{CD}] &= \frac{1}{2} (\eta_{AC} M^\pm_{DB} + \eta_{BD} M^\pm_{CA} - \eta_{BC} M^\pm_{DA} - \eta_{AD} M^\pm_{CB}), \\ [M^\pm_{AB}, Q^\pm_\alpha] &= \frac{1}{2} (\gamma_{AB})_{\alpha\beta} Q^\pm_\beta, \\ \{Q^\pm_\alpha, Q^\pm_\beta\} &= -\frac{1}{2} (\gamma^{AB} C)_{\alpha\beta} M^\pm_{AB}. \end{aligned} \quad (12)$$

As far as the Hodge duality (11) is a projection, it is transmitted to the connections in (5),  $\mathbf{A}^A_i M^\pm_{\mathcal{A}} = \mathbf{A}^{\pm A}_i M^\pm_{\mathcal{A}}$ . Conversely, we can start with a Chern-Simons action constructed with (anti)self-dual connections, that is

$$L^\pm_{SCS} = STr \int_X d^3x \varepsilon^{ijk} \left( \mathbf{A}^\pm_i \partial_j \mathbf{A}^\pm_k + \frac{2}{3} \mathbf{A}^\pm_i \mathbf{A}^\pm_j \mathbf{A}^\pm_k \right), \quad (13)$$

where  $\mathbf{A}^\pm_i = \mathbf{A}^{\pm A}_i M_{\mathcal{A}} = \mathbf{A}^{\pm A}_i M^\pm_{\mathcal{A}}$ , thus,  $L^\pm_{SCS}$  and  $L^\mp_{SCS}$  are the actions of the corresponding factors in (9) and the result can be obtained from (8) if we substitute  $\mathbf{A}$  by  $\mathbf{A}^\pm$

$$\begin{aligned} L^\pm_{SCS} &= \int_X d^3x \varepsilon^{ijk} \left( A^{\pm AB}_i \partial_j A^{\pm}_{kBA} + \frac{2}{3} A^{\pm}_{iA}{}^B A^{\pm}_{jB}{}^C A^{\pm}_{kC}{}^A \right) \\ &+ \int_X d^3x \varepsilon^{ijk} \left( A^{\pm\alpha}_i \partial_j A^{\pm\beta}_k C_{\alpha\beta} + \frac{1}{2} (\gamma_{AB} C)_{\alpha\beta} A^{\pm AB}_i A^{\pm\alpha}_j A^{\pm\beta}_k \right). \end{aligned} \quad (14)$$

In order to compare this action with the Chern-Simons supergravity action, as usual, we identify the three-dimensional connection with  $\omega_i^{ab}$ , where  $a, b = 1, 2, 3$ , and the dreibein  $e^a_i = A^{0a}_i$ , in such a way that the corresponding Minkowski metric will be  $\eta^{ab} = \text{diag}(1, -1, -1)$ .

For the fermionic sector we have the identification  $A_i^\alpha = \psi_i^\alpha$ . By this identification, we obtain the action

$$L_{SCS} = \int_X d^3x \varepsilon^{ijk} \left( -\omega_i^a \partial_j \omega_{ka} - e_i^a \partial_j e_{ka} + \psi_i^\alpha \partial_j \psi_k^\beta C_{\alpha\beta} + \frac{1}{3} \varepsilon^{abc} \omega_{ia} \omega_{jb} \omega_{kc} \right. \\ \left. + e_i^a e_j^b \omega_{kab} - \frac{1}{4} (\gamma_{ab} C)_{\alpha\beta} \omega_i^{ab} \psi_j^\alpha \psi_k^\beta - \frac{1}{2} (\gamma_{0a} C)_{\alpha\beta} e_i^a \psi_j^\alpha \psi_k^\beta \right), \quad (15)$$

where  $\omega_i^{ab} = \epsilon^{abc} \omega_{ic}$ .

From this result it is easy to construct the corresponding (anti)self-dual actions. In order to do that, we choose the supersymmetric charges  $Q_\alpha$  to be Majorana such that the anti-self-dual and self-dual sectors will have the same supersymmetries

$$Q_\alpha^+ = \begin{pmatrix} 0 \\ -(\varepsilon Q)_\alpha \end{pmatrix}, \quad Q_\alpha^- = \begin{pmatrix} Q_\alpha \\ 0 \end{pmatrix}, \quad (\alpha = 1, 2). \quad (16)$$

Further we have that  $\omega_i^{\pm a} = \frac{1}{2}(\omega_i^a \pm e_i^a)$ . We obtain

$$L_{SCS}^+ = \int_X d^3x \varepsilon^{ijk} \left( -2\omega_i^{+a} \partial_j \omega_{ka}^+ + \omega_i^{+\alpha} \partial_j \omega_k^{+\beta} C_{\alpha\beta} + \frac{4}{3} \varepsilon^{abc} \omega_{ia}^+ \omega_{jb}^+ \omega_{kc}^+ \right. \\ \left. - \frac{1}{2} [(\gamma_{0a} + \frac{1}{2} \varepsilon_{abc} \gamma^{bc}) C]_{\alpha\beta} \omega_i^{+a} \psi_j^{+\alpha} \psi_k^{+\beta} \right) \\ = \int_X d^3x \varepsilon^{ijk} \left( -\omega_i^a \partial_j \omega_{ka} - e_i^a \partial_j e_{ka} - \omega_i^a \partial_j e_{ka} - e_i^a \partial_j \omega_{ka} + \frac{1}{2} \psi_i^T \partial_j \psi_k \right. \\ \left. + \frac{1}{6} \varepsilon^{abc} (\omega_{ia} \omega_{jb} \omega_{kc} + e_{ia} e_{jb} e_{kc} + 3\omega_{ia} \omega_{jb} e_{kc} + 3\omega_{ia} e_{jb} e_{kc}) \right. \\ \left. - \frac{1}{4} (\omega_i^a + e_i^a) \psi_j^T \varepsilon \tau_a \psi_k \right), \quad (17)$$

which coincides with the sum of the three-dimensional “standard” and “exotic” supergravity.

### III. CHERN-SIMONS SUPERGRAVITY DUAL ACTION

Now we want to show that a “dual” action to the (2 + 1)-Chern-Simons supergravity action (13) can be constructed following [18,8]. We consider first the action

$$L_{SCS} = \int_X d^3x \varepsilon^{ijk} \eta_{AB} \mathbf{A}_i^A \mathbf{H}_{jk}^B, \quad (18)$$

where  $\eta_{AB}$  is given by Eq. (6) and  $\mathbf{H}_{ij}^A = H_{ij}^{AB} M_{AB} + H_{ij}^\alpha Q_\alpha$  with  $H_{ij}^{AB}$

$$H_{ijAB} = \partial_i A_{jAB} + \frac{1}{3} f_{AB CDEF} A_i^{CD} A_j^{EF} + \frac{1}{3} f_{AB\alpha\beta} A_i^\alpha A_j^\beta \quad (19)$$

and  $H_{ij}^\alpha$  is given by

$$H_{ij}^\alpha = \partial_i A_j^\alpha + f_{AB\beta}^\alpha A_i^{AB} A_j^\beta. \quad (20)$$

Now, as usual we propose a parent action in order to derive the dual action to (4),

$$L_D = \int_X d^3x \varepsilon^{ijk} \left( a \mathbf{B}_i^A \mathbf{H}_{jkA} + b \mathbf{A}_i^A \mathbf{G}_{jkA} + c \mathbf{B}_i^A \mathbf{G}_{jkA} \right), \quad (21)$$

where  $a, b, c$  are the coupling constants and  $\mathbf{B}_i^A$  and  $\mathbf{G}_{jk}^A$  are  $\text{Osp}(2, 2|1)$ -Lie algebra valued Lagrange multipliers.

It is a straightforward matter to show that Eq. (4) can be derived from the parent action. To see that, consider first the partition function of the parent action of the form

$$Z = \int \mathcal{D}\mathbf{A} \mathcal{D}\mathbf{G} \mathcal{D}\mathbf{B} \exp(+iL_D). \quad (22)$$

Integration with respect to the Lagrange multiplier fields  $\mathbf{G}$  and  $\mathbf{B}$  define the Lagrangian  $L_D^*$  as following  $Z = \int \mathcal{D}\mathbf{A} \exp(+iL_D^*)$  where

$$\exp(+iL_D^*) = \int \mathcal{D}\mathbf{G} \mathcal{D}\mathbf{B} \exp\left(+i \int_X d^3x \varepsilon^{ijk} (a \mathbf{B}_i^A \mathbf{H}_{jkA} + b \mathbf{A}_i^A \mathbf{G}_{jkA} + c \mathbf{B}_i^A \mathbf{G}_{jkA})\right). \quad (23)$$

One can integrate out first with respect  $\mathbf{G}$ . This gives

$$\exp(+iL_D^*) = \int \mathcal{D}\mathbf{B} \delta(b \mathbf{A}_i^A + c \mathbf{B}_i^A) \exp\left(+ia \int_X d^3x \varepsilon^{ijk} \mathbf{B}_i^A \mathbf{H}_{jkA}\right). \quad (24)$$

Further integration with respect to  $\mathbf{B}$  gives the final form

$$L_D^* = -\frac{ab}{c} \int_X d^3x \varepsilon^{ijk} \mathbf{A}_i^A \left( \partial_j \mathbf{A}_{kA} + \frac{1}{3} f_{ABC} \mathbf{A}_j^B \mathbf{A}_k^C \right). \quad (25)$$

A choice of the constants of the form

$$c = -\frac{4\pi}{g}, \quad a = b = 1, \quad (26)$$

immediately gives the original Lagrangian (4).

The “dual” action  $\tilde{L}_D^*$  can be computed as usually in the euclidean partition function, by integrating first out with respect to the physical degrees of freedom  $\mathbf{A}$ . We can of course expand the index  $\mathcal{A}$ , in its fermionic and bosonic parts, and the result is the same,

$$\exp\left(-\tilde{L}_D^*\right) = \int \mathcal{D}\mathbf{A} \exp\left(-L_D\right). \quad (27)$$

The resulting action is of the gaussian type in the variable  $\mathbf{A}$  and thus after some computations using supergroup techniques (see appendix [12]) it is easy to find the “dual” action

$$\tilde{L}_D^* = \int_X d^3x \varepsilon^{ijk} \left\{ -\frac{3}{4a} (a \partial_i \mathbf{B}_{j\mathcal{A}} + b \mathbf{G}_{ij\mathcal{A}}) [\mathbf{M}^{-1}]_{kn}^{\mathcal{AC}} \varepsilon^{lmn} (a \partial_l \mathbf{B}_{m\mathcal{C}} + b \mathbf{G}_{lm\mathcal{C}}) + c \mathbf{A}_i^{\mathcal{A}} \mathbf{G}_{jk\mathcal{A}} \right\}, \quad (28)$$

where  $[\mathbf{M}]$  is given by  $[\mathbf{M}]_{\mathcal{AB}}^{ij} = \varepsilon^{ijk} f^{\mathcal{C}}_{\mathcal{AB}} B_{k\mathcal{C}}$  whose inverse is defined by  $[\mathbf{M}]_{\mathcal{AB}}^{ij} [\mathbf{M}^{-1}]_{jk}^{\mathcal{BC}} = \delta_k^i \delta_{\mathcal{A}}^{\mathcal{C}}$ .

The partition function is finally given by

$$Z = \int \mathcal{D}\mathbf{G} \mathcal{D}\mathbf{B} \sqrt{S \det(\mathbf{M}^{-1})} \exp(-\tilde{L}_D^*), \quad (29)$$

where  $Sdet$  is the superdeterminant [17].

In this “dual action” the  $\mathbf{G}$  field is not dynamical and can be integrated, if we take the values (26) for the constants of the parent action (21), then the integration of this auxiliary field gives

$$Z = \int \mathcal{D}\mathbf{B} \exp \left\{ -\frac{4\pi}{g} \varepsilon^{lmn} \left( \mathbf{B}_l^{\mathcal{A}} \partial_m \mathbf{B}_{n\mathcal{A}} - \frac{4\pi}{g} f_{\mathcal{ABC}} \mathbf{B}_l^{\mathcal{A}} \mathbf{B}_m^{\mathcal{B}} \mathbf{B}_n^{\mathcal{C}} \right) \right\}. \quad (30)$$

This action is the Chern-Simons action for the Lagrange multiplier  $Osp(2, 2|1)$ -valued one form field  $\mathbf{B}$ . It can be observed that the coupling constant  $g$  has been inverted *i.e.*  $g \leftrightarrow -\frac{1}{g}$ .

#### IV. CONCLUDING REMARKS

In this paper we have shown that the gravitational duality in  $(2+1)$ -Chern-Simons supergravity constitutes a new example of the idea of *superduality* [13,14]. We have found also that this example is  $S$ -self-dual because the dual theory comes described by the same type of action.



It is known from Ref. [7] that for the abelian case, Chern-Simons duality is related to *mirror symmetry*. For the nonabelian case, it is well known that the Jones invariants of links are defined for long value of the coupling constant  $g$  while the Vassiliev invariants are defined for small  $g$ . Strong/weak coupling duality of nonabelian Chern-Simons theory would be relevant to connect them in a new way.

On the other hand in [14] an non-trivial example of superduality interchanging chiral and twisted multiplets was worked out. It is very well known that this feature is precisely another realization of mirror symmetry. Thus the global picture seems to be consistent. It will be of interest to investigate the possible relation of the "*gravitational (super) duality*" of this paper and Refs. [8,11–14] to that which arise considering gravitational branes in type IIA superstrings and M-theory [19]. Another interesting problem concerns the application of non-abelian Chern-Simons duality to the Chern-Simons (super) string theory constructed in Ref. [20]. Here Chern-Simons duality may be useful to find another dual realizations to the Chern-Simons (super) string theory. Some of this work will be reported in [21].

### Acknowledgments

This work was supported in part by CONACyT grants 28454E and 33951E.

## REFERENCES

- [1] E. Alvarez, L. Alvarez-Gaumé, and Y. Lozano, “An Introduction to T-duality in String Theory”, Nucl. Phys. Proc. Suppl. **41** (1995) 1-20, hep-th/9410237.
- [2] E. Alvarez, L. Alvarez-Gaumé, and Y. Lozano, “On Non-abelian Duality”, Nucl. Phys. B **424** (1994) 155, hep-th/9403155.
- [3] F. Quevedo, “Duality and Global Symmetries”, Nucl. Phys. Proc. Suppl. **61** A (1998) 23.
- [4] E. Witten, Commun. Math. Phys. **121** (1989) 351.
- [5] A.P. Balachandran, L. Chandar and B. Sathiapalan, Int. J. Mod. Phys. A **11** (1996) 3587 .
- [6] B.P. Dolan, J. Phys. A **32** (1999) L243; Nucl. Phys. B **554** (1999) 487; “Duality in the Quantum Hall Effect: the Role of Electron Spin”, cond-mat/0002228.
- [7] A. Kapustin and M.J. Strassler, “On Mirror Symmetry in Three Dimensional Abelian Gauge Theories”, JHEP **9904**:021 1999, hep-th/9902033.
- [8] H. García-Compeán, O. Obregón, C. Ramírez and M. Sabido, “Remarks on  $2 + 1$  Self-dual Chern-Simons Gravity”, Phys. Rev. D, **61** (2000) 0850022-1.
- [9] A. Achúcarro and P.K. Townsend, Phys. Lett. B **180** (1986) 89.
- [10] E. Witten, Nucl. Phys. B **311** (1988) 46.
- [11] H. García-Compeán, O. Obregón, J.F. Plebanski and C. Ramírez, “Towards a gravitational analogue to S duality in non-Abelian gauge theories”, Phys. Rev. D, **57** (2000) 7501.
- [12] H. García-Compeán, O. Obregón and C. Ramírez, “Gravitational Duality in MacDowell-Mansouri Gauge Theory”, Phys. Rev. D, **58** (1998) 104012.

- [13] H. García-Compeán, J.A. Nieto, O. Obregón and C. Ramírez, “Dual description of supergravity MacDowell-Mansouri theory”, Phys. Rev. D, **59** (1999) 124003-1.
- [14] C.P. Burgess, A.T. Grisaru, M. Kamela, M.E. Knutt-Wehlau, P. Page, F. Quevedo and M. Zbarjad, “Spacetime Duality and Superduality”, Nucl. Phys. B **542** (1999) 195, hep-th/9809085.
- [15] K. Koehler, F. Mansouri, C. Vaz and L. Witten , Mod. Phys. Lett. A **5** (1990) 935; J. Math. Phys. **32** (1991) 239; P. Valtancoli, “Exact Results in Chern-Simons Supergravity”, hep-th/9711172.
- [16] J. A. Nieto, J. Socorro and O. Obregón, Phys. Rev. Lett. **76**, 3482 (1996).
- [17] B. DeWitt, *Supermanifolds*, Second edition (Cambridge University Press, Cambridge, England, 1992).
- [18] O. Ganor and J. Sonnenschein, Int. J. Mod. Phys. A **11** (1996) 5701; N. Mohammadi, hep-th/9507040; Y. Lozano, Phys. Lett. B **364** (1995) 19.
- [19] C.M. Hull, Nucl. Phys. **B509** (1998) 216.
- [20] E. Witten, “Chern-Simons Gauge Theory as a String Theory”, hep-th/9207094.
- [21] H. García-Compeán, O. Obregón and C. Ramírez, “On a Vector Nonabelian Chern-Simons Duality”, work in progress.